Advanced Mechanics: Final Exam

Semester Ib 2022-2023

January 24th, 2023

Please note the following rules:

- you are not allowed to use the book or the lecture notes, nor other notes or books
- you have 2 hours to complete the exam
- please write your student number on each paper sheet you hand in
- please use a separate paper sheet for each problem
- please raise your hand for more paper or to ask a question
- some useful equations are provided on the next page
- it is important that you show your work

Points for each problem:

- Problem 1: *15 points*
- Problem 2: *30 points*
- Problem 3: *20 points*
- Problem 4: *35 points*

Useful equations

- Principal moments of inertia:

$$I_{xx} = \int dm\,(y^2 + z^2)\,, \tag{0.1}$$

$$I_{yy} = \int dm\,(x^2 + z^2)\,, \tag{0.2}$$

$$I_{zz} = \int dm\,(x^2 + y^2)\,. \tag{0.3}$$

- Products of inertia:

$$I_{xy} = -\int dm\,x\,y\,, \qquad I_{xz} = -\int dm\,x\,z\,, \qquad I_{yz} = -\int dm\,y\,z\,. \tag{0.4}$$

- Moment of inertia of a uniform cylinder of mass M and radius R with respect to its axis:

$$I = \frac{1}{2}MR^2 \tag{0.5}$$

- Center of mass for N point masses m_i

$$\vec{R} = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{j=1}^{N} m_j} \tag{0.6}$$

- Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0\,. \tag{0.7}$$

- Modified Lagrange equations containing Lagrange multipliers (f_k being the constraints):

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_k \lambda_k \frac{\partial f_k}{\partial q_j} = 0\,. \tag{0.8}$$

- Generalised force:

$$Q_j = \sum_k \lambda_k \frac{\partial f_k}{\partial q_j}\,. \tag{0.9}$$

- Generalised momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{0.10}$$

- Hamiltonian:

$$H(q_j, p_j; t) = \dot{q}_i p_i - L(q_j, \dot{q}_j; t) \tag{0.11}$$

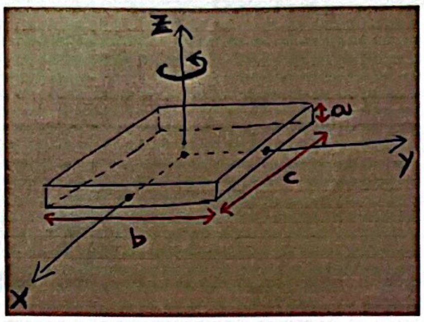

Figure 1.

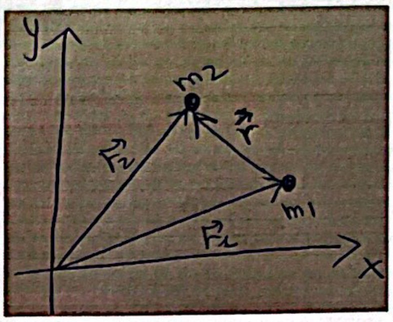

Figure 2.

Problem 1

Find the principal moments of inertia for a book of uniformly distributed mass M and sides (a, b, c) as in Fig. 1, rotating about its centre. Is any of the products of inertia non-zero? No
[15 points]

$I_{xx} = \frac{M}{12}(a^2+b^2)$

$I_{yy} = \frac{M}{12}(a^2+c^2)$

$I_{zz} = \frac{M}{12}(b^2+c^2)$

Problem 2

Consider two point masses m_1 and m_2 moving in a plane (assume no friction) and interacting with each other by means of a potential $V(r) = \frac{1}{2}kr^2$, r being the distance between them (Fig. 2).

(2a) Starting from the Lagrangian for the two point masses in terms of their respective coordinates $\vec{r}_1$ and $\vec{r}_2$, obtain the Lagrangian in terms of the center of mass ($\vec{R}$) and relative position ($\vec{r}$).
[10 points]

$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$

(2b) Derive the equations of motion for the coordinates $\vec{R} = (X, Y)$ and $\vec{r} = (x, y)$. What is the frequency of the relative motion?
[10 points]

$M\ddot{X} = 0$ $\quad \mu\ddot{x} = -kx$ $\quad \omega = \sqrt{\frac{k}{\mu}}$

$M\ddot{Y} = 0$ $\quad \mu\ddot{y} = -ky$

(2c) Compute the generalised momenta and from there obtain the Hamiltonian. Then verify explicitly that the Hamiltonian is equal to the total energy.
[10 points]

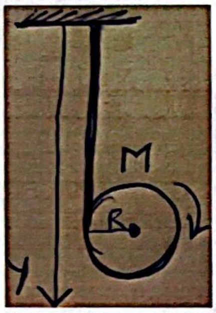

Figure 3.

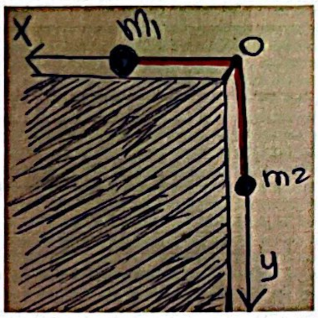

Figure 4.

Problem 3

Consider the object in Fig. 3: a massless string is suspended vertically from a fixed point and wrapped around a uniform cylinder of mass M and radius R. The cylinder moves vertically down due to gravity, rotating as the string unwinds. [*The moment of inertia of the cylinder is provided in the formula sheet*].

(3a) Derive the Lagrangian, using y as your generalised coordinate. $\frac{3}{4}M\dot{y}^2 + Mgy$
[15 points]

(3b) Obtain the equation of motion and verify that the cylinder accelerates downwards with acceleration equal to $2\,g/3$.
[5 points]

Problem 4

Consider the system represented in Fig. 4, consisting of two point masses, m_1 and m_2, connected by an inextensible string passing over a frictionless massless pulley. The first mass is free to move on a frictionless horizontal table, while the second mass moves vertically. Use coordinates x and y equal to the distances of, respectively, m_1 and m_2 from the pulley.

(4a) Compute the Lagrangian in x and y coordinates. $\frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + m_2 g y$
[10 points]

$-l + x + y = 0$

(4b) Use the modified Lagrange equations to derive the generalised force.
[15 points] $Q = m_1\ddot{x} = m_2(\ddot{y}+g)$

(4c) Using the Newtonian approach, verify that the generalised force derived in (4b) is equal to the the tension of the string on the two masses.
[10 points] $Q = -T$